On The Existence of the Solution for Fractional Integro-Differential Inclusion with Integral Boundary Conditions

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الملخص

خلال هذا البحث قدمنا شروطاً كافية لوجود الحل لصنف من مسائل القيم الحدودية المؤلفة من مضمنات تكاملية تفاضلية من رتبة كسرية مع شروط حدودية تكاملية في حالة الدوال المحدبة وغير المحدبة، بالأعتماد على مبرهنات النقطة الثابتة.

ABSTRACT

In this paper, we establish sufficient conditions for the existence of solutions for a class of boundary value problem of fractional integro-differential inclusions and nonlinear integral conditions, in the cases of convex and nonconvex valued using fixed point theorems.

1. Introduction

Differential inclusion is a generalization of the concept of ordinary differential equation, its arise in many situations including differential variational inequalities, projected dynamical systems, dynamic Coulomb friction problems and fuzzy set arithmetic. Differential equations of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. There are numerous applications to problems in viscoelasticity, electrochemistry, control, porous media, electromagnetics, etc.

The existence of the solutions of differential inclusion was studded in many works, see [2],[3],[6],[9]. in [6] the authors studded the existence of the solutions of the boundary value problem with fractional order differential inclusion and nonlinear integral conditions of the form:

$$^{c}D^{\alpha}y(t) \in F(t, y(t))$$

$$y(0) - y'(0) = \int_{0}^{T} p_1(s, y(s))ds$$

$$y(T) + y'(T) = \int_0^T p_2(s, y(s))ds$$

In this paper we consider the following fractional Integro-differential boundary value problem with integral boundary condition

$$^cD^\alpha y(t) \in F(t, y(t), \int_0^T k_1(t, s, y(s))ds, \int_0^t k_2(t, s, y(s))ds) \quad (1)$$

$$ay(0) - by'(0) = \int_0^T p_1(s, y(s))ds$$

$$cy(T) + dy'(T) = \int_0^T p_2(s, y(s))ds \quad (2)$$

where a, b, c, d are constants , $ab + cd \neq 0$, $^CD^\alpha$is the standard Caputo derivative,$1 < \alpha < 2$, $t \in J := [0, T]$, $k_1, k_2 : J \times J \times R \to R$, $p_1, p_2 : J \times R \to R$ are Carath´eodory functions, $F: J \times R \times R \times R \to P(R)$ is a multivalued function.

Here, for brevity let

$$K_1y(t) = \int_0^T k_1(t, s, y(s))ds \quad , K_2y(t) = \int_0^t k_2(t, s, y(s))ds$$

2. Preliminaries.

In this section, we introduce notations, definitions, and preliminary facts from set-valued analysis which are used throughout this paper. For further background and details pertaining to this section we refer the reader in fractional calculus to [8] and in multivalued function to [1], [5] and [7].

Let $C[J, R]$ denotes the Banach space of all continuous functions from J into R, normed

$\|u\| = sup\{|u(t)| : t \in J\}$

and $L^1[J, R]$ denotes the Banach space of measurable functions $u : J \to R$ which are Lebesgue integrable, normed by

$$\|u\|_L = \int_0^T |u(t)|\, dt,$$

Let $(X, |.|)$ be a normed space, and $P(X)$ be the family of all nonempty subsets of X.

$P_{cl}(X) = \{Y \in P(X) : Y \text{ is closed}\}$,

$P_b(X) = \{Y \in P(X) : Y \text{ is bounded}\}$,

$P_{cp}(X) = \{Y \in P(X)\ Y \text{ is compact}\}$,

$P_c(X) = \{Y \in P(X) : Y \text{ is convex}\}$,

$P_{cl,c}(X) = \{Y \in P(X)\ Y \text{ is closed and convex}\}$,

$P_{cp,c}(X) = \{Y \in P(X) : Y \text{ is compact and convex}\}$.

A set- valued function $F : X \to P(X)$ is called *convex* (*closed*) valued if $F(x)$ is convex (closed) for all $x \in X$. F is called *bounded* valued on bounded sets if $F(B) = \cup_{x \in B} F(x)$ is bounded in X for all

$B \in P_b(X)$, F is called *upper semi-continuous* (*u.s.c*) on X if the set $F^{-1}(G) = \{x \in X : F(x) \subset A\}$ is open in X for every open set G in X. F is called *lower semi-continuous* (*l.s.c*) on X if the set $F^{-1}(E) = \{x \in X : F(x) \subset E\}$ is closed in X for every closed set E in X. F is called continuous if it is lower as well as upper semi-continuous on X. The mapping F has a fixed point if there is $x \in X$ such that $x \in F(x)$. The set of fixed points of the multivalued operator F will be denoted by $Fix(F)$. A set-valued function $F : J \to P(R)$ is said to be *measurable* if for any $x \in X$, the function $t \mapsto d(x,F(t)) = inf\{|x-u| : u \in F(t)\}$ is measurable.

The following definitions are used in the sequel.

Definition 2.1. A set-valued function $F : J \times R \to P(R)$ is said to be *Carath'eodory* if:
(i) $t \to F(t,u)$ is measurable for each $u \in R$,
(ii) $u \to F(t,u)$ is *u.s.c.* for almost $t \in J$.

For each $y \in C(J,R)$, define the set of selections for F by

$$S_F(y) = \{h \in L^1(J,R) : h(t) \in F(t,y(t)) \ a.e. \ t \in J\}.$$

Let (X,d) be a metric space induced from the normed space $(X,|\cdot|)$. Consider

$H_d : P(X) \times P(X) \to R^+ \cup \{\infty\}$ given by

$$H_d(A,B) = max(\sup_{a\in A} d(a,B), \sup_{b\in B} d(B,a))$$

where $d(A,b) = inf_{a\in A} d(a,b)$ and $d(a,B) = inf_{b\in B} d(a,b)$.
Then $(P_{b,cl}(X), H_d)$ is a metric space and $(P_{cl}(X), H_d)$ is a generalized metric space.

Definition 2.2. A set-valued function $F : J \times R \to P_{cl}(R)$ is called
(i) $l(t)-Lipschitz$ if there exists $l \in L^1(J,R^+)$ such that
$H(F(t,x), F(t,y)) \le l(t)\|x-y\|$, for each $x, y \to X$
(ii) a *contraction* if it is $l(t)-Lipschitz$ with $\|l\| < 1$.

Definition 2.3. A multi-valued map $F : X \to P\ (X)$ is:
(a) *compact* if its range $F(X)$ is relatively compact in X, i.e., $\overline{F(X)}$ is compact in X;
(b) *locally compact* if every point $x \in X$ has a neighborhood $V(x)$ such that the restriction of F to $V(x)$ is compact;

Remark: it is clear that (a) $\to$ (b).

Lemma 2.1. [7] Let $F: X \to P(Y)$ be a closed locally compact multimap. Then F is *u.s.c.* .

Lemma 2.2. (Bohnerblust-Karlin, [4]). Let X be a Banach space, B a nonempty subset of X, which is bounded, closed, and convex. Suppose

$F: B \to P(X)\backslash\{0\}$ is $u.s.c.$ with closed, convex values, and such that $G(B) \subset B$ and $\overline{G(B)}$ compact. Then F has a fixed point.

Lemma 2.3. (Covitz-Nadler)[6] Let (X,d) be a complete metric space. If $G: X \to P_{cl}(X)$ is a Contraction, then G has a fixed point.
For completeness, in this section, we mainly demonstrate and study the definitions and some fundamental

Definition 2.4. [8]Let $\alpha > 0$, for a functiony : $(0,+\infty) \to R$. The the fractional integral of order α of y is defined by

$$I^{\alpha}y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds$$

Provided the integral exists.

Definition 2.5. The Caputo derivative of a function $y: (0,+\infty) \to R$ is given by

$$^{c}D^{\alpha}y(t) = I^{n-\alpha}(D^{n}y(t)) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}y^{(n)}(s)ds$$

Provided the right side is point wise defined on $(0,+\infty)$, where $n = [\alpha] + 1$, and $[\alpha]$ denotes the integer part of the real numberα.

Lemma 2.4. .[8] Let $\alpha > 0$. Then the differential equation

$$^{c}D^{\alpha}h(t) = 0$$

has the solution

$$h(t) = +c_0 + c_1 t + c_2 t^2 + c_3 t^3 + \cdots + c_{n-1}t^{n-1}$$

Where $c_i \in R, i = 0,1,2,\ldots.n-1$ and $n = [\alpha] + 1$.

Lemma 2.5. [8] Let $\alpha > 0$; Then

$$I^{\alpha}\, {^{c}D^{\alpha}}h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + c_3 t^3 + \cdots + c_{n-1}t^{n-1}$$

for some $c_i \in R, i = 0,1,2,\ldots.n-1$ and $n = [\alpha] + 1$.

Lemma 2.6. Let $1 < \alpha \le 2$ and let $h \in L^1(J,R)$ and $p_1, p_2 : J \to R$ be continuous functions. a function y is a solution of the fractional BVP

$$^{c}D^{\alpha}y(t) = h(t) \quad (3)$$

$$ay(0) - by'(0) = \int_0^T p_1(s)ds \quad (4)$$

$$cy(T) + dy'(T) = \int_0^T p_2(s)ds \quad (5)$$

If and only if y is a solution of the fractional integral equation

$$y(t) = P(t) + \int_0^T G(t,s)h(s)ds \quad (6)$$

where

$$P(t) = \frac{1}{acT + ad + bc}\left((c(T-t) + d)\int_0^T p_1(s)ds + (at+b)\int_0^T p_2(s)ds \right)$$

and

$G(s,t)$

$$= \begin{cases} \dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \dfrac{(at+b)}{acT+ad+bc}\left(\dfrac{c(T-s)^{\alpha-1}}{\Gamma(\alpha)} + \dfrac{d(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}\right) & 0 \le s \le t \\ -\dfrac{(at+b)}{acT+ad+bc}\left(\dfrac{c(T-s)^{\alpha-1}}{\Gamma(\alpha)} + \dfrac{d(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}\right) & t < s \le T \end{cases}$$

Proof. Assume that y satisfies (3); then Lemma (2.5) implies

$$y'(t) = c_1 + \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)\,ds \tag{7}$$

$$y(t) = c_o + c_1 t + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds \tag{8}$$

from (4) and (5), we have

$$ac_o - bc_1 = \int_0^T p_1(s)ds \tag{9}$$

$$cc_o + (cT+d)c_1 + c\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds + d\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds = \int_0^T p_2(s)ds \tag{10}$$

by solving (9) – (10), we obtain

$$c_o = \frac{1}{acT+ad+bc}\left[(cT+d)\int_0^T p_1(s)ds + b\int_0^T p_2(s)ds - b\left(c\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds + d\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds\right)\right] \tag{11}$$

$$c_1 = \frac{1}{acT+ad+bc}\left[-c\int_0^T p_1(s)ds + a\int_0^T p_2(s)ds - a\left(c\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds + d\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds\right)\right] \tag{12}$$

From (8), (11), (12) and the fact that $\int_0^T = \int_0^t + \int_t^T$, we obtain (6). Conversely, if y satisfies equation (6), then clearly (3)-(5) hold. ■

Note that if $a = b = c = d = 1$ then lemma(2.6) will give the lemma(3.4) of [6], which is a special case of our lemma.

Definition 2.6. A function $y \in AC^1(J,R)$ is said to be a solution of (1)–(2), if there exists a function $h \in L^1(J,R)$ with $h(t) \in F(t, y(t), K_1y(t), K_2y(t))$, for a.e. $t \in J$, such that

${}^cD^\alpha y(t) = h(t) \quad a.e. \ t \in J$

and the function y satisfies conditions (2).

3. Main results

I) The convex case

Theorem 3.1. Assume that the following assumptions hold

(H1) $F : J \times R \times R \times R \to P_{cp,c}(R)$ is a Caratheodory multi-valued map; *i.e.*

(i) $t \to F(t,x,y,z)$ is measurable for each $(x,y,z) \in R^3$

(ii) $(x,y,z) \to F(t,x,y,z)$ is *u.s.c.* for almost $t \in J$.

(H2) There exist $q_1, q_2 \in L^1(J,R^+)$ such that

$$|k_i(t,y))| \le q_i(t)|y| \quad (i = 1,2) \quad \textit{for all } t \in J,\ y \in R.$$

(H3) There exist $\phi \in L^1(J,R^+)$ and a continuous function $\psi: [0,\infty) \times [0,\infty) \times [0,\infty) \to [0,\infty)$ which is a non decreasing in either of its independent variables and such that

$$\|F(t,x,y,z)\|_P = sup\{|h| : h \in F(t,x,y,z)\} \le \phi(t)\psi(|x|,|y|,|z|)$$

for all $t \in J$ *and* $(x,y,z) \in R^3$;

(H4) There exist $\phi_1, \phi_2 \in L^1(J,R^+)$ and a continuous nondecreasing functions $\psi_1, \psi_2 : [0,\infty) \to [0,\infty)$ such that

$$|p_i(t,y)| \le \phi_i(t)\, \psi_i(|y|) \quad (i = 1,2) \ \textit{for all } t \in J,\ y \in R.$$

(H5) There exists a number $M > 0$ such that

$$M > \frac{1}{acT + ad + bc}\left((cT + d)\,\psi_1(M)\int_0^T |\phi_1(s)|ds + (aT + b)\,\psi_2(M)\int_0^T |\phi_2(s)|ds \right) + \psi(M, Q_1M, Q_2M)\int_0^T G(t,s)\phi(s)(1 + q_1(s) + q_2(s))ds$$

where $Q_i = \int_0^T q_i(s)ds \quad (i = 1,2)$, and $G(t,s)$ as in lemma 2.6, then the BVP (1)–(2) has at least one solution on J.

Proof: we transform problem (1)-(2) into fixed point problem by considering the multivalued operator $N(y): C(J,R) \to P(C(J,R))$ as

$$N(y) := \left\{ v \in C(J,R) : v(t) = P(t,y(t)) + \int_0^T G(t,s)h(s)ds \ ; h \in S_F(y) \right\}$$

where $S_F(y) := \{h \in L(J,R) : h(t) \in F\big(t, y(t), K_1y(t), K_2y(t)\big) \ a.e.\ t \in J\}$ and

$$P(t,y(t)) = \frac{1}{acT + ad + bc}\left((c(T - t) + d)\int_0^T p_1(s,y(s))ds + (at + b)\int_0^T p_2(y(s))ds \right)$$

and the function $G(t,s)$ is given above. Clearly, from Lemma 2.6, the fixed points of N are solutions to (6) for some $h \in S_F$. We shall show that N satisfies the assumptions of the Bohnerblust-Karlin theorem (lemma 2.2.) The proof will be given in several steps.

Step 1: $N(y)$ is convex for each $y \in C(J,R)$. Indeed, if v_1 and v_2 belong to $N(y)$, then there exist $h_1, h_2 \in S_F(y)$ such that, for all $t \in J$, we have

$$v_i(t) = P(t, y(t)) + \int_0^T G(t,s)h_i(s)\,ds \qquad i = 1,2$$

Let $0 < \lambda < 1$. Then, for each $t \in J$, we have

$$(\lambda v_1 + (1-\lambda)v_2)(t) = P(t,y(t)) + \int_0^T G(t,s)\,[\lambda h_1(s) + (1-\lambda)h_2(s)]\,ds.$$

Since $S_F(y)$ is convex (because F has convex values), we have $(\lambda v_1 + (1-\lambda)v_2)(t) \in N(y)$.

Step 2: N maps bounded set to itself in $C(J,R)$. Let $B_r = \{y \in C(J,R) : \|y\| \le r\}$ then B_r is a bounded closed convex set in $C(J,R)$. We shall prove that there exists a positive number r^* such that $N(B_{r^*}) \subset B_{r^*}$, let $y \in B_r$ for some $r > 0$. Then for each $v \in N(y)$ and $t \in J$, from **(H2)–(H4)**, we have:

$$|v(t)| = \left| \frac{1}{acT + ad + bc}\left((c(T-t)+d)\int_0^T p_1(s,y(s))\,ds + (at+b)\int_0^T p_2(s,y(s))\,ds \right) + \int_0^T G(t,s)h(s)\,ds \right|$$

for some $h \in S_F(y)$

$$|v(t)| \le \frac{1}{acT + ad + bc}\left((c(T-t)+d)\int_0^T |p_1(s,y(s))|\,ds + (at+b)\int_0^T |p_2(s,y(s))|\,ds \right) + \int_0^T G(t,s)|h(s)|\,ds$$

$$|v(t)| \le \frac{1}{acT + ad + bc}\left((cT+d)\,\psi_1(|y|)\int_0^T \phi_1(s)\,ds + (aT+b)\,\psi_2(|y|)\int_0^T \phi_2(s)\,ds \right) + \psi\left(|y|, \int_0^T |k_1(s,y(s))|\,ds, \int_0^t |k_2(s,y(s))|\,ds \right)\int_0^T G(t,s)\phi(s)\,ds$$

$$\|v\| \leq \frac{1}{acT+ad+bc}\left((cT+d)\,\psi_1(r)\int_0^T \phi_1(s)ds + (aT+b)\,\psi_2(r)\int_0^T \phi_2(s)\,ds \right) + \psi(r,Q_1r,Q_2r)\int_0^T G(t,s)\phi(s)ds$$

Now from **(H5)** there exists a positive number M such that

$$\|v\| \leq \frac{1}{acT+ad+bc}\left((cT+d)\,\psi_1(M)\int_0^T \phi_1(s)ds + (aT+b)\,\psi_2(M)\int_0^T \phi_2(s)ds \right) + \psi(M,Q_1M,Q_2M)\int_0^T G(t,s)\phi(s)ds < M$$

and hence $N(B_M) \subset B_M$

Step 3: N maps bounded sets into equicontinuous sets of $C(J,R)$.
Let $t_1,t_2 \in J$
with $t_1 < t_2$, let B_r be a bounded set in $C(J,R)$ as in Step 2, and let $y \in B_r$ then for each $v \in N(y)$. We have

$$|v(t_2)-v(t_1)| = \left| P(t_2,y(t_2)) + \int_0^T G(t_2,s)h(s)ds - P(t_1,y(t_1)) - \int_0^T G(t_1,s)h(s)ds \right|$$

$$|v(t_2)-v(t_1)| \leq \frac{1}{acT+ad+bc}\left[c(t_2-t_1)\int_0^T |p_1(s,y(s))|ds + a(t_2-t_1)\int_0^T |p_2(s,y(s))|ds \right] + \int_0^T |G(t_2,s)-G(t_1,s)||h(s)|ds$$

$$= \frac{1}{acT + ad + bc}(t_2 - t_1)\left(c\int_0^T |p_1(s,y(s))|ds \ + a\int_0^T |p_2(s,y(s))|ds \right)$$

$$+ \int_0^T |G(t_2,s) - G(t_1,s)||h(s)|ds$$

since $G(t,s)$ is continuous in t, we conclude that as $t_1 \to t_2$, the right-hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3 together with the Arzel´a-Ascoli theorem, we claim that $N : C(J,R) \to P(C(J,R))$ is a compact multivalued map.

Step 4: $N(y)$ is closed for each $y \in C(J,R)$. Let $\{v_n\}_{n\geq 0} \in N(y)$ be such that $v_n \longrightarrow v_*$ *as* $n \longrightarrow \infty$ in $C(J,R)$. Then, $v_* \in C(J,R)$ and there exist $h_n \in S_F(y)$, such that for each $t \in J$,

$$v_n(t) = P(t,y(t)) + \int_0^T G(t,s)h_n(s)ds$$

From the fact that F has compact values, we shall pass to a subsequence if necessary to obtain that h_{n_m} converges weakly to h_* in $L^1(J,R)$ and therefore $h_* \in S_F(y)$, then we have for each $t \in J$,

$$v_{n_m}(t) \longrightarrow v_*(t) = P(t,y(t)) + \int_0^T G(t,s)h_*(s)ds$$

thus, $v_* \in N(y)$.

Since N closed and compact multi-valued map. We conclude from Lemma 2.1 that F is *u.s.c.* .Hence, we conclude that N is a compact multi-valued map, u.s.c. with convex closed values on the bounded closed convex set B_M and $N(B_M) \subset B_M$ .In view of Lemma 2.2, we deduce that N has a fixed point which is a solution to problem (1)-(2). ■

II) The non convex case

Theorem 3.2. Assume that the following assumptions hold :

(H6) $F : J \times R \times R \times R \to P_{cp}(R)$ has the property that $F(\cdot,x,y,z) : J \to P_{cp}(R)$ is measurable, and integrably bounded for each $(x,y,z) \in R^3$.

(H7) There exists $l \in L^1(J,R^+)$ such that

$$H_d(F(t,x_1,x_2,x_3),F(t,y_1,y_2,y_3)) \leq l(t)(|x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|) \quad \forall x_i, y_i \in R, \ (i = 1,2,3)$$

$d(0,F(t,0,0,0)) \leq l(t)$ $a.e.$ $t \in J$.

(H8) There exists $N_1, N_2 > 0$ such that

$$|k_i(t,x) - k_i(t,y)| \leq N_i|x - y| \quad (i = 1,2), \ \forall x,y \in R,$$

(H9) There exists $l_i \in L^1(J,R^+)$ such that

$$|p_i(t,x) - p_i(t,y)| \leq l_i(t)|x - y| \quad (i = 1,2) \ \forall x,y \in R,$$

If

$$\gamma := \frac{1}{acT+ad+bc}\left((cT+d)\int_0^T l_1(s)\,ds \; + (aT+b)\int_0^T l_2(s)\,ds \right)$$

$$+\int_0^T (1+TN_1+TN_2)\, G(t,s) l(s)\, ds \; < 1$$

then the BVP (1)–(2) has at least one solution on J.

Remark: For each $y \in C(J,R)$, the set $S_F(y)$ is nonempty since, by (H6), F has a measurable selection (see [5], Theorem III.6).

Proof: We shall show that N satisfies the assumptions of the Covitz-Nadler Lemma 2.3 The proof will be given in two steps.

***Step 1*:** $N(y) \in P_{cl}(C(J,R))$. As step 4 in theorem 3.1 we can conclude that $N(y)$ is closed . and hence $N(y) \in P_{cl}(C(J,R))$.

***Step* 2:** $N(y)$ is a contraction multi valued mapping. Then we have to prove the existence of a constant $0 < \gamma < 1$ such that

$H_d(N(x), N(y)) \le \gamma\|x - y\| \quad \forall x, y \in C(J,R)$

Let $x, y \in C(J,R)$ and $v_1 \in N(x)$. Then, there exists $h_1 \in F(t,x)$ such that, for each $t \in J$,

$$v_1(t) = P(t) + \int_0^T G(t,s)h_1(s)ds$$

From **(H7)** it follows that

$$H_d(F(t,x(t),K_1x(t),K_2x(t)), F(t,y(t),K_1y(t),K_2y(t))) \le (1+TN_1+TN_2)\; l(t)\,|x(t)-y(t)|$$

Hence, for each $t \in J$ there exists $w \in F(t,y(t))$ such that

$$|h_1(t) - w| \le (1+TN_1+TN_2)\; l(t)\,|x(t)-y(t)|$$

Consider $U: J \to P(R)$ given by

$U(t) = \{w \in R: |h_1(t) - w| \le (1+TN_1+TN_2)\, l(t)|x(t)-y(t)|.\}$

Since the multivalued operator $V(t) = U(t) \cap F(t,y(t))$ is measurable (see [5], Proposition III.4), there exists a function $h_2(t)$ which is a measurable selection for V . Thus, for each $t \in J$, we have $h_2(t) \in F(t,y(t))$, and

$$|h_1(t) - h_2(t)| \le (1+TN_1+TN_2)\; l(t)\,|x(t)-y(t)|$$

Now for each $t \in J$, define $v_2(t) = P(t,y(t)) + \int_0^T G(t,s)h_2(s)ds$

Then,

$$|v_1(t) - v_2(t)| = \left|P(t,x(t)) + \int_0^T G(t,s)h_1(s)ds - P(t,y(t)) - \int_0^T G(t,s)h_2(s)ds\right|$$

$$\le \frac{1}{acT+ad+bc}\left[(c(T-t)+d)\int_0^T |p_1(s,x(s)) - p_1(s,y(s))|ds \right.$$

$$\left. + (at+b)\int_0^T |p_2(s,x(s)) - p_2(s,y(s))|ds \right]$$

$$+\int_0^T G(t,s)|h_1(s) - h_2(s)|ds$$

$$|v_1(t) - v_2(t)| \le \frac{\|x-y\|}{acT+ad+bc}\left[(cT+d)\int_0^T l_1(s)\,ds + (aT+b)\int_0^T l_2(s)\,ds\right] + \int_0^T G(t,s)l(s)(|x(s)-y(s)|, |K_1x(s)-K_1y(s)|, |K_2x(s) - K_2y(s)|)\,ds$$

Therefore,

$$\|v_1 - v_2\| \le \frac{\|x-y\|}{acT+ad+bc}\left[(cT+d)\int_0^T l_1(s)\,ds + (aT-b)\int_0^T l_2(s)\,ds\right] + \|x-y\|\int_0^T (1+TN_1+TN_2)\,G(t,s)l(s)\,ds = \gamma\|x-y\|$$

By an analogous relation, obtained by interchanging the roles of x and y, it follows that

$H_d(N(x), N(y)) \le \gamma\|x-y\|$

Therefore, N is a contraction, and so by Lemma 2.3, N has a fixed point y that is a solution to (1.1)–(1.3). The proof is now complete. ■

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